The Lévy-Itô Decomposition Theorem

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Abstract

This a free translation with additional explanations of *Processus à Accroissement Inde*pendants Chapitre I: La Décomposition de Paul Lévy, by J.L. Bretagnolle, in Ecole d'Eté de Probabilités, Lecture Notes in Mathematics 307, Springer 1973. The Lévy-Khintchine representation of infinitely divisible distributions is obtained as a by-product.

As this proof makes use of martingale methods, it is pedagogically more suitable for students of financial mathematics than some other approaches. It is hoped that the end notes will also help to make the proof more accessible to this audience.

1 Lévy processes

Definition 1.1 Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t)$ be a filtered probability space. A stochastic process $X = (X_t)_t$ with values in \mathbb{R}^n is said to be an *n*-dimensional Lévy process if

- (a) X is adapted to $(\mathcal{F}_t)_t$ (i.e. X_t is \mathcal{F}_t -measurable for all $t \geq 0$).
- (b) $X_0 = 0$ a.s.
- (c) $X_{t+s} X_t$ is independent of \mathcal{F}_t , and has the same law as X_s , for all $s, t \geq 0$.
- (d) $(X_t)_t$ is continuous in probability.

Consequences: Put $\varphi_t(u) := \mathbb{E}[e^{i\langle u, X_t \rangle}]$, where $\langle u, v \rangle$ represents the inner product on \mathbb{R}^n . From (d), the $\varphi_t(u)$ are continuous in the pair (t, u). From (b), $\varphi_0(u) = 1$. From (c), $\varphi_{t+s}(u) = \varphi_t(u)\varphi_s(u)$, so $\varphi_t(u) \neq 0$ for any (t, u). One can thus write $\varphi_t(u) = e^{-t\psi(u)}$, where ψ is a continuous function null at 0.

Conversely, if ψ is a continuous function null at 0, and such that for all $t \geq 0$, $\varphi_t(u) := e^{-t\psi(u)}$ is positive definite (which means that for each choice of finitely many u_j, λ_j we have $\sum_{i,j} \lambda_i \bar{\lambda_j} \varphi_t(u_i - u_j) \geq 0^1$), then, by Bochner's theorem, $\varphi_t(u)$ is the Fourier transform of a probability measure on \mathbb{R}^n . One can therefore construct a projective family of measures on $(\mathbb{R}^n)^{\mathbb{R}^+}$ by the formula

$$\mathbb{E}[e^{i\langle u, X_{t_1}\rangle + \dots + i\langle u, X_{t_n}\rangle}] = \mathbb{E}[e^{i\langle u_1 + \dots + u_n, X_{t_1}\rangle + \dots + i\langle u_n, X_{t_n} - X_{t_{n-1}}\rangle}]$$

$$= \varphi_{t_1}(u_1 + \dots + u_n) \cdot \varphi_{t_2 - t_1}(u_2 + \dots + u_n) \dots \varphi_{t_n - t_{n-1}}(u_n)$$

for all finite choices of $0 \le t_1 \le \cdots \le t_n$. The process $(X_t)_t$ on $(\mathbb{R}^n)^{\mathbb{R}^+}$, given by Kolmogorov's Theorem, clearly adapted to the natural filtration $\mathcal{F}_t := \sigma(X_s : s \le t)$, possesses the properties

(a), (b), (c), (d), because the fact that $X_{t+s} - X_t \to 0$ in probability when $s \downarrow 0$ is an immediate consequence² of the fact that $\varphi_s(u) \to 1$ when $s \downarrow 0$. Thus there is a Lévy process to each ψ possessing those properties.

Theorem 1.2 Suppose that S is a countable subset of \mathbb{R}^+ . Then there exists a \mathbb{P} -null set N such that on N^c the map $t \mapsto X_t$ has left- and right limits along S (làg, làd). If one defines $Y_t := \lim_{s \in S, s \downarrow t} X_s$ on N^c , and 0 on N, then Y is adapted to $\overline{\mathcal{F}}_t$, where $\overline{\mathcal{F}}_t$ is the completion of \mathcal{F}_t in \mathcal{F} , i.e. completed by all the null sets of \mathcal{F} (or $\bigvee_t \mathcal{F}_t$). Moreover, Y is càdlàg (continue à droite, pourvu limites à gauche). Finally, Y is a modification of X, i.e. for all t, $\mathbb{P}(X_t \neq Y_t) = 0$.

Proof: Suppose that $u \in \mathbb{Q}^n$, and that M_t^u is defined by $M_t^u := \frac{e^{i\langle u, X_t \rangle}}{\varphi_t(u)}$. For each u, $(M_t^u)_t$ is a (complex) martingale, thus, save for a null set N^u , làglàd along S (see for example Neveu p. 129-132)³ Left and right limits along S therefore exist simultaneously for all the M^u , except for the null set $N := \bigcup_{u \in \mathbb{Q}^n} N^u$. Suppose that, for $u \in N^c$, the map $u \in N^c$ could have two distinct cluster points $u \in N^c$ as $u \in \mathbb{Q}^n$ such that $u \in \mathbb{Q}^n$ such that u

Consequence: If we have a Lévy process in the sense of Definition 1.1, one can now take for X the regularised (i.e. càdlàg) version Y, and for \mathcal{F}_t one can take $\overline{\sigma(X_s:s\leq t)}$. We now study a fixed Lévy process X (if one exists), and also fix $(\mathcal{F}_t)_t$ for the remainder of this chapter.

Theorem 1.3 (0-1 Law) If we define $\mathcal{F}_{t^+} := \bigcap_{s>t} \mathcal{F}_s$, then $\mathcal{F}_{t^+} = \mathcal{F}_t$

Proof: \mathcal{F}_{t+} may be considered as a countable intersection, since $\mathcal{F}_u \subseteq \mathcal{F}_v$ when $u \leq v$. Thus if $t_1 \leq t_2$, we have $\mathbb{E}[e^{i\langle u, X_{t_1}\rangle}|\mathcal{F}_{t_2}] = \mathbb{E}[e^{i\langle u, X_{t_1}\rangle}|\mathcal{F}_{t_2^+}]$, a common version being $e^{i\langle u, X_{t_1}\rangle}$. If $t_1 > t_2$, then⁶

$$\mathbb{E}[e^{i\langle u, X_{t_1} \rangle} | \mathcal{F}_{t_2^+}] \stackrel{\text{a.s.}}{=} \lim_{s \downarrow t_2} \mathbb{E}[e^{i\langle u, X_{t_1} \rangle} | \mathcal{F}_s]$$

$$\stackrel{\text{a.s.}}{=} \lim_{s \downarrow t_2} e^{i\langle u, X_s \rangle} \varphi_{t_1 - s}(u)$$

$$\stackrel{\text{a.s.}}{=} e^{i\langle u, X_{t_2} \rangle} \varphi_{t_1 - t_2}(u)$$

$$\stackrel{\text{a.s.}}{=} \mathbb{E}[e^{i\langle u, X_{t_1} \rangle} | \mathcal{F}_{t_2}]$$

Thus for all u, s we have $\mathbb{E}[e^{i\langle u, X_s \rangle} | \mathcal{F}_{t+}] \stackrel{\text{a.s.}}{=} \mathbb{E}[e^{i\langle u, X_s \rangle} | \mathcal{F}_t]$. The two conditional expectations are equal a.s. for all random variables $e^{i\langle u, X_s \rangle}$, and hence for all $\nabla_t \mathcal{F}_t$ -measurable random variables.

Henceforth, therefore, $(\mathcal{F}_t)_t$ is right–continuous: $\mathcal{F}_{t^+} = \mathcal{F}_t$. (In particular, $A \in \mathcal{F}_0 = \mathcal{F}_{0^+}$ implies $\mathbb{P}(A) = 0$ or 1.)

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Theorem 1.4 (Strong Markov property)

If T is a stopping time, then on $\{T < \infty\}$ the process $(X_{T+t} - X_T)_{t \ge 0}$ is a Lévy process with the same law as X, adapted to $(\mathcal{F}_{T+t})_t$, càdlàg, and independent of \mathcal{F}_T .

Proof: Suppose first that T is bounded, and let $A \in \mathcal{F}_T$. Take $u_i \in \mathbb{Q}^n$, and $t \in \mathbb{R}^+$. Then

$$\mathbb{E}\left[I_A e^{i\sum_j \langle u_j, X_{T+t_j} - X_{T+t_{j-1}} \rangle}\right] = \mathbb{P}(A) \prod_j \varphi_{t_j - t_{j-1}}(u_j)$$

on application of the optional sampling theorem to the martingales $M_t^{u_j}$.⁸ If T is not bounded, the formula remains true when applied to $T \wedge n$ and $A \cap \{T \leq n\} \in \mathcal{F}_{T \wedge n}$. One can pass to the limit by dominated convergence, and hence the formula is true without restrictions. On the one hand it shows that $X_{T+t} - X_T$ is independent of \mathcal{F}_T , and on the other that $X_{T+t} - X_T$ has properties (a), (b), (c). It is clear that $X_{T+t} - X_T$ is càdlàg, thus a fortiori continuous in probability.⁹

Corollary 1.5 A Lévy process which has amplitudes of discontinuities a.s. bounded has moments of all orders.

Proof: Suppose M is such that $\mathbb{P}(\exists t: |X_t - X_{t-}| \geq M) = 0$. Put $T_1 := \inf\{t | |X_t| \geq M\}$, and $T_n := \inf\{t: t > T_{n-1}, |X_t| \geq M\}$. The right-continuity implies that the T_n form a strictly increasing sequence of stopping times. Since $|X_T - X_{T-}| \leq M$ for all T, by iduction we have $\sup_{s \leq T_n} |X_s| \leq 2nM$, the strong Markov property implies that $T_n - T_{n-1}$ is independent of $\mathcal{F}_{T_{n-1}}$, with the same law as T_1 , and hence $\mathbb{E}[e^{-T_n}] = \mathbb{E}[e^{-T_1}]^n = a^n$, where a < 1. Thus $\mathbb{P}(|X_t| \geq 2nM) \leq \mathbb{P}(T_n < t) \leq a^n$, and hence there exists an exponential moment for X_t .

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2 Poisson Process

This is an increasing adapted Lévy process which grows only by jumps of amplitude 1. We will denote it by $(N_t)_t$ in what follows, with or without supplementary indices.

If $T_1 := \inf\{t : N_t \neq 0\}$, then $\{T_1 > t\} = \{N_t = 0\}$. T_1 is a stopping time, $\mathbb{P}(T_1 > t + s) = \mathbb{P}(N_{t+s} - N_t = 0, N_t = 0)$, so by the strong Markov property, $\mathbb{P}(T_1 > t + s) = \mathbb{P}(T_1 > s)\mathbb{P}(T_1 > t)$. This function being decreasing and bounded, we have $\mathbb{P}(T_1 > t) = e^{-at}$ for some $a \in \mathbb{R}^+$ ($T_1 > 0$ a.s.). For a = 0, $N_t \equiv 0$; if not, T_1 is a.s. finite, and if we put $T_n - T_{n-1} := \inf\{t > 0 : N_{t+T_n} - N_{t+T_{n-1}} > 0\}$, then $T_n - T_{n-1}$ is independent of $\mathcal{F}_{T_{n-1}}$ and has the same law as T_1 .

Then $\mathbb{P}(N_t = n) = \mathbb{P}(T_{n+1} > t, T_n \leq t) = \frac{a^n t^n}{n!} e^{-at}$, 11, and $\mathbb{E}[e^{iuN_t}] = e^{-at(1-e^{iu})}$. As this function is positive definite 12, then by the *converse* on p. 1, there exists a Poisson process 13 Finally (by Corollary 1.5), $\hat{N}_t := N_t - at$ and $(N_t - at)^2 - at$ are integrable, and are martingales, as one immediately verifies 14

Theorem 2.1 If M is a centered square–integrable martingale, N a Poisson process, then for all t,

$$\mathbb{E}[M_t N_t] = \mathbb{E}\left[\sum_{n \ge 0} (M_{T_n} - M_{T_n-}) I_{\{T_n \le t\}}\right]$$

(where T_n are the jump times of N.)

Proof: Suppose that $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ is a partition of [0, t]. By using the martingale property of M_t and $\hat{N}_t := N_t - at$ repetitively, we obtain:

$$\mathbb{E}[M_t N_t] = \mathbb{E}[M_t \hat{N}_t] = \mathbb{E}\left[\sum_i (M_{t_{i+1}} - M_{t_i}) \sum_j (\hat{N}_{t_{j+1}} - \hat{N}_{t_j})\right]$$

$$= \mathbb{E}\left[\sum_i (M_{t_{i+1}} - M_{t_i})(\hat{N}_{t_{i+1}} - \hat{N}_{t_i})\right] = \mathbb{E}\left[\sum_i (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i})\right]$$

If the step size $\sup_{i}(t_{i+1}-t_i)$ tends to 0,

$$\sum_{i} (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}) \stackrel{\mathbb{P} \text{ or a.s.}}{\longrightarrow} \sum_{n>0} (M_{T_n} - M_{T_n-}) I_{\{T_n \le t\}}$$

The proof is complete if one can show that the Lebesgue dominated convergence theorem is applicable: Now

$$\left| \sum_{i} (M_{t_{i+1}} - M_{t_i}) (\hat{N}_{t_{i+1}} - \hat{N}_{t_i}) \right| \le 2 \sup_{s \le t} |M_s| \cdot N_t$$

and both factors are in L^2 ($\mathbb{E}[\sup_{s \leq t} |M_s|^2] \leq 4\mathbb{E}[|M_t|^2]$) ¹⁵.

3 The Decomposition of Paul Lévy

3.1 Jump Measure

Let B be a Borel set in \mathbb{R}^n with $0 \notin \overline{B}$. By recursion, we define the stopping times

$$S_B^1 := \inf\{t > 0 : X_t - X_{t-} \in B\}$$
 $S_B^n := \inf\{t > S_B^{n-1} : X_t - X_{t-} \in B\}$

One easily verifies that , because of right–continuity, $X(t,\omega)$ is jointly measurable in (t,ω) , hence the S^n_B are stopping times adapted to \mathcal{F}_{t+} , thus to \mathcal{F}_t , by the 0-1 law. ¹⁶ The right-continuity implies that $S^1_B>0$ a.s. and that $N_t(B):=\sum\limits_{n\geq 0}I_{\{S^n_B\leq t\}}<\infty$ a.s. (If not, there would be a

discontinuity of the second kind on the trajectory X_t^{17} .) $N_t(B)$ is therefore a Poisson process (see later), and we denote by $\nu(B)$ the parameter $\mathbb{E}[N_1(B)]$.¹⁸ For each ω , $N_t(dx,\omega)$ defines a σ -finite measure on $\mathbb{R}^n - \{0\}$, thus $\nu(dx) := \mathbb{E}[N_1(dx)]$ is equally a σ -finite measure (≥ 0) on $\mathbb{R}^n - \{0\}$.

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3.2 Associated Jump Processes

Lemma 3.1 Let f be bounded measurable on B in \mathbb{R}^p . Then

$$\int_{B} f(x) N_{t}(dx) = \sum_{n>1} f(X_{S_{B}^{n}} - X_{S_{B}^{n}}) I_{\{S_{B}^{n} \le t\}}$$

Proof: If f is a step function, $f = \sum_j a_j I_{B_j}$ where $\sum_j I_{B_j} = I_B$, the integral is $\sum_j a_j N_t(B_j) = \sum_j a_j (\sum_n I_{\{S_{B_j}^n \le t\}})$. But the family $\{S_B^n\}$ is the union of the $\{S_{B_j}^n\}$, and the result follows for step functions f. Else, one approximates f uniformly by step functions... 20

Remark 3.2 In fact, for B a Borel set $(0 \notin \overline{B})$ of course, it suffices that f be finite everywhere on B for the formula to be true, for $N_t(B)$ is finite a.s. for all t. In particular, we denote by $X_t(B)$ the quantity

$$\int_{N} x \ N_{t}(dx) = \sum_{n \ge 1} (X_{S_{B}^{n}} - X_{S_{B}^{n}}) I_{\{S_{B}^{n} \le t\}}$$

Lemma 3.3 $\int_B f(x) N_t(dx)$, $X_t(B)$ are Lévy processes adapted to $(\mathcal{F}_t)_t$.

Proof: $N_t(dx)$ is an adapted Lévy process!

Lemma 3.4 $X_t - X_t(B)$ is a Lévy process adapted to $(\mathcal{F}_t)_t$.

[To demonstrate that $N_t(B), X_t(B)$ and $X_t - X_t(B)$ are adapted Lévy processes, we note that conditions (a) and (b) are automatically satisfied, and that (d) follows from the càdlàg property²¹. Only (c) remains to be verified. Now let Z_t be one of the above-mentioned processes. Note that $Z_{t+s} - Z_t \in \sigma(X_u : u \leq t \leq t + s)$ is independent of \mathcal{F}_t . The same goes for the stationarity of the increments²²...]

 $X_t - \int_{\{|x| \ge 1\}} N_t(dx)$ does not have jumps of |amplitude| ≥ 1 (by Lemma 3.1), is a Lévy process adapted to $(\mathcal{F}_t)_t$ (by Lemma 3.4), and can thus be centered by a translation γt (by Corollary 1.5). We may therefore restrict ourselves to the study of:

3.3 The Lévy Decomposition for centered Lévy process with jumps bounded by 1

Lemma 3.5 Suppose that $B \subseteq \{|x| \ge 1\}$ is such that $0 \notin \overline{B}$, and that $f : \mathbb{R}^n \to \mathbb{R}$ is such that fI_B is in $L^2(\nu(dx))$ (the jump measure $\nu(dx)$ has been introduced in §3.1). Then we have

$$\mathbb{E}\left[\int_{B} f(x) \ N_{t}(dx)\right] = t \int_{B} f(x) \ \nu(dx)$$

and

$$\mathbb{E}\left[\left(\int_{B} f(x) N_{t}(dx) - t \int_{B} f(x) \nu(dx)\right)^{2}\right] = t \int_{B} f(x)^{2} \nu(dx)$$

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Proof: If f is a step function, $f = \sum_{i} a_{i} I_{B_{i}}$, we have²³

$$\mathbb{E}\left[\sum_{j} a_{j} N_{t}(B_{j})\right] = \sum_{j} a_{j} \mathbb{E}[N_{t}(B_{j})] = t \sum_{j} a_{j} \nu(B_{j})$$

For the second equation, note that if $B_i \cap B_j = \emptyset$, then by Theorem 2.1 we have $\mathbb{E}[\hat{N}_t(B_i)\hat{N}_t(B_j)] = 0^{24}$. For f not a step function, choose a sequence of step functions f_n such that $f_n I_B$ tends to $f I_B$ in $L^2(d\nu)$, and thus also in $L^1(d\nu)$. We then have convergence of the corresponding stochastic integrals in $L^2(d\mathbb{P})$ and $L^1(d\mathbb{P})$.

We now introduce \mathcal{M} , the space of càdlàg centered square-integrable martingales on $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to $(\mathcal{F}_t)_t$. We equip this space with the topology (of Fréchet) induced by the family of seminorms $q_t(M) := \mathbb{E}[M_t^2]^{\frac{1}{2}}$. From the classical inequality²⁶ $\mathbb{E}[\sup_{s \leq t} M_s^2] \leq 4\mathbb{E}[M_t^2]$ we deduce

that the q_t -convergence of a sequence implies, with probability 1, the uniform convergence of the trajectories on the interval [0, t], and thus the limit is càdlàg. q_t -convergence equally implies convergence of the random variables in L^2 , and thus preserves the centeredness and martingale properties. In other words, \mathcal{M} is closed in its topology.

Lemma 3.6 If B is as above, and

$$\mathcal{H}_B := \left\{ \int_B f(x) \ N_t(dx) - t \int_B f(x) \ \nu(dx) : fI_B \in_2 (d\nu) \right\}$$

then \mathcal{H}_B is a closed subspace of \mathcal{M} .

Proof: We have 27

$$t||fI_B||_{L^2(d\nu)}^2 = q_t(M_{fI_B})^2 \tag{*}$$

where $M_{fI_B,t} := \int_B f(x) N_t(dx) - t \int_B f(x) \nu(dx) : fI_B \in_2 (d\nu)$.

- (α): For fI_B a step function, M_{fI_B} is a martingale in \mathcal{M} , because to each Poisson process N corresponds the martingale $\hat{N}_t := N_t \mathbb{E}[N_t]$ in \mathcal{M} . As all $L^2(d\nu)$ -functions are limits of step functions, M_{fI_B} is a martingale in \mathcal{M} as soon as $fI_B \in L^2(d\nu)$.
- (β): Now \mathcal{H}_B is closed in \mathcal{M} , because q_t -convergence also implies convergence in $L^2(d\nu)$, by $(\star)^{28}$

Lemma 3.7 Let B be as in Lemma 3.5. If $M \in \mathcal{M}$ is continuous at the times S_B^n , then M is orthogonal to \mathcal{H}_B .

Proof: By Theorem 2.1, for all $A \subseteq B$ and all t we have $\mathbb{E}[M_t N_t(A)] = 0$. Now, the $\{N_t(A) | A \subseteq B\}$ generate \mathcal{H}_B^{29} .

Corollary 3.8 If B_1 , B_2 are disjoint Borel sets, with $0 \notin \bar{B_1} \cup \bar{B_2}$, then the processes $(X_t(B_1))_t$ and $(X_t(B_2))_t$ are independent Lévy processes.

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Proof: That they are Lévy processes has already been demonstrated (Lemma 3.5). If now

$$M_{1,t}^u := \frac{e^{i\langle u, X_t(B_1)\rangle}}{\mathbb{E}[e^{i\langle u, X_t(B_1)\rangle}]} - 1 \qquad M_{2,t}^v := \frac{e^{i\langle v, X_t(B_2)\rangle}}{\mathbb{E}[e^{i\langle v, X_t(B_2)\rangle}]} - 1$$

then these two martingales are orthogonal, by Lemma 3.7. We have $\forall s,t \in \mathbb{R}^+ \ \forall u,v \in \mathbb{R}^n \ (\mathbb{E}[M^u_{1,t}M^v_{2,s}]=0)$, which ensures³⁰ their independence.

Now put

$$Y_t(B) := X_t(B) - \mathbb{E}[X_t(B)] = X_t(B) - t \int_B x \ \nu(dx)$$

Then Y is both a Lévy process and a martingale in \mathcal{M} . If we define

$$B_k := \left\{ \frac{1}{k+1} < |x| \le \frac{1}{k} \right\}$$
 and $A_n := \bigcup_{k=1}^n B_k$

then the $Y(B_k)$ are pairwise independent, and $X - Y(A_n)$ and $Y(A_n)$ are orthogonal, and even independent (retrace the proof of Corollary 3.8). Consequently, the series (sum) of the $Y(B_k)$ converges in L^2 , and thus in \mathcal{M} , to a Lévy process X^d , while $X - Y(A_n)$ converges to a Lévy process X^c in \mathcal{M} .

Thus 32 :

Lemma 3.9 $X_t = X_t^c + X_t^d$, where X^c is a martingale with continuous sample paths, and

$$X_t^d := \int_{|x| \le 1} x \left(N_t(dx) - t\nu(dx) \right)$$

Remark 3.10 This last integral exists in L^2 , and hence $\int_{\{|x| \le 1\}} |x|^2 \nu(dx) < \infty$.

It remains to characterize the continuous part: We will show that it is necessarily Gaussian Lévy process, i.e. that each X_t^c is Gaussian. For this, it suffices to show this for each one–dimensional projection (a well-known property of the Gaussians³³). In other words, we must show that³⁴:

Lemma 3.11 Let B_t be a one-dimensional centered Lévy process with continuous sample paths. Then there is $\sigma^2 \in \mathbb{R}^+$ such that

$$\mathbb{E}[e^{iuB_t}] = e^{-\frac{1}{2}u^2\sigma^2t}$$

Proof: A Lévy process without discontinuities has moments of all orders, by Corollary 1.5. If $\mathbb{E}[B_t^2] = 0$ for all t > 0, then the problem is solved. If not, we can assume that $\mathbb{E}[B - t^2] = t$, by multiplying the process by a constant. Note that $\mathbb{E}[B_t^4] = at + bt^2 + ct^3$: It suffices to put $\mathbb{E}[e^{iuB_t}] = e^{-t\psi(u)}$, to differentiate four times at the origin $(\psi(u))$ is of the class \mathcal{C}^{∞} , like $\varphi_t(u)$ and to observe that $\psi'(0) = 0.35$ Now let $P := \{0 = t_0 < t_1 < \cdots < t_n = t\}$ be a partition of

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[0,t] whose step size $\sup_{j}\{(t_{j+1}-t_{j})\}$ will tend to 0. We denote by Δt_{j} the quantity $t_{j+1}-t_{j}$, and by ΔB_{j} the quantity $B_{t_{j+1}}-B_{t_{j}}$. We then have

$$\mathbb{E}[e^{iuB_t} - 1] = \mathbb{E}\left[\sum_j e^{iuB_{t_{j+1}}} - e^{iuB_{t_j}}\right]$$

$$= iu\sum_j \mathbb{E}\left[e^{iuB_{t_j}}\Delta B_j\right] - \frac{1}{2}u^2\sum_j \mathbb{E}\left[e^{iuB_{t_j}}(\Delta B_j)^2\right]$$

$$- \frac{1}{2}u^2\sum_j \mathbb{E}\left[(\Delta B_j)^2 \cdot (e^{iu(B_{t_j} + \theta_j \Delta B_j)} - e^{iuB_{t_j}})\right]$$

where the θ_j are numbers between 0 and 1, by the second order Taylor formula. In the second line, the first term is zero: ΔB_j has zero expectation, and is independent of B_{t_j} .

The second term equates to $-\frac{1}{2}u^2\sum_j \varphi_{t_j}(u)\Delta t_j$, and thus tends to $-\frac{1}{2}u^2\int_0^t e^{-s\psi(u)} ds$ as the step size tends to 0.

The third term tends to 0: Let A_{α} be the event

$$A_{\alpha} := \left\{ \sup_{j} \sup_{t_j \le u, v \le t_{j+1}} |B_u - B_v| < \alpha \right\}$$

The third term can therefore be bounded by ³⁶

$$\frac{1}{2}|u|^3 \int_{A_\alpha} \alpha(\sum_j \Delta B_j^2) \ d\mathbb{P} + |u|^2 \int_{A_\alpha^c} \sum_j \Delta B_j^2 \ d\mathbb{P}$$

 $thus^{37}$ by

$$\tfrac{1}{2}\alpha|u|^3\mathbb{E}[\sum_j\Delta B_j^2]+|u|^2\sqrt{\mathbb{P}(A_\alpha^c)}\cdot\sqrt{\mathbb{E}[(\sum_j\Delta B_j^2)^2]}$$

thus, taking into account the evaluation of $\mathbb{E}[B_t^4]$, by $\frac{1}{2}\alpha|u|^3t+u^2\sqrt{\mathbb{P}(A_\alpha^c)}(O(t+t^3))^{\frac{1}{2}}$. Finally, note that that the continuity of the sample paths implies that, as the partition gets finer, $\mathbb{P}(A_\alpha^c) \to 0$. The expectation of the third term has limit $\leq \frac{1}{2}\alpha|u|^3t$, and thus converges to zero. We therefor obtain the equation:

$$e^{-t\psi(u)} - 1 = -\frac{1}{2}u^2 \int_0^t e^{-s\psi(u)} ds$$

which identifies $\psi(u)$ with $\frac{1}{2}u^2$.³⁸

3.4 The Decomposition Theorem

Theorem 3.12 (A) Let X be an n-dimensional Lévy process. Then

$$X_{t} = B_{t} + t\mathbb{E}\left[X_{1} - \int_{|x|>1} x N_{1}(dx)\right] + \int_{\{|x|\geq1\}} x N_{t}(dx) + \int_{\{|x|<1\}} x \left(N_{t}(dx) - t\nu(dx)\right)$$

where

 \dashv

- B_t is a centered Gaussian Lévy process with a.s. continuous sample paths.
- $N_t(dx)$ is a family of Poisson processes, independent of B_t , with $N_t(A)$ idependent of $N_t(B)$ if $A \cap B = \emptyset$, and with $\nu(dx) = \mathbb{E}[N_1(dx)]$.
- $\nu(dx)$ is a positive measure on $\mathbb{R}^n \{0\}$, with $\int |x|^2 \wedge 1 \nu(dx) < \infty$.
- The first stochastic integral is in the sense of L^0 and the second in the sense L^2 .
- (B) Formula for the law³⁹: Under these conditions,

$$\psi(u) = -\frac{1}{t} \mathbb{E}[e^{i\langle u, X_t \rangle}]$$

$$= \frac{1}{2} Q(u) - i\langle a, u \rangle + \int_{\{|x| \ge 1\}} 1 - e^{i\langle u, x \rangle} \nu(dx) + \int_{|x| < 1\}} 1 - e^{i\langle u, x \rangle} + i\langle u, x \rangle \nu(dx)$$

where Q is a positive definite quadratic form⁴⁰ on \mathbb{R}^n , $a \in \mathbb{R}^n$, and $\nu(dx)$ is as in (A).

- (C) Conversely, given Q, a, ν as in (B), here exists a Lévy process whose law is given by the formula in (B).
- (D) The representation in (B) is unique.

Proof: To summarize the preceding, (B) is obvious. (D) The uniqueness of the decomposition is clear by constructions. For the law, put

$$\psi(u) := \frac{1}{2}Q_j(u) - i\langle a_j, u \rangle + \int_{\{|x| \ge 1\}} (1 - e^{i\langle u, x \rangle} \nu_j(dx) + \int_{\{|x| < 1\}} 1 - e^{i\langle u, x \rangle} + i\langle u, x \rangle \nu_j(dx)$$

where Q_j, a_j, ν_j are as in (B), and j = 1, 2. Let y be a unit vector in \mathbb{R}^n :x $\lim_{s \to \infty} \frac{\psi(sy)}{s^2} = \frac{1}{2}Q_j(y)$, for $\lim_{s\to\infty}\frac{\langle a,sy\rangle}{s^2}=0$, and the same for the integral terms, by dominated convergence⁴¹ 42. ψ therefore determines Q. We will determine all the projections of ν (let us suppose that the Lévy process is one-dimensional)⁴³:

$$\psi(u) - \frac{1}{2}Q(u) - \frac{1}{2}\int_{u-1}^{u+1} \psi(v) - \frac{1}{2}Q(v) \ dv = -\int e^{iux} (1 - \frac{\sin x}{x}) \ \nu(dx)$$

The lefthand side therefore determines (Bochner) the (positive) measure $(1 - \frac{\sin x}{x}) \nu(dx)$, and thus ν , because $(1 - \frac{\sin x}{x}) > 0$ on $\mathbb{R} - \{0\}$. Now a can be identified by subtraction. (C): For each t, $e^{-tQ(u)}$, $e^{it\langle a,u\rangle}$ and $e^{-t(1-e^{i\langle u,x\rangle})}$ are positive definite⁴⁴, thus the equation (B) defines⁴⁵ a continuous function if $\int |x|^2 \wedge 1 \nu(dx) < \infty$, with each $e^{-t\psi(u)}$ being positive definite, and thus, according to the observations following the first paragraph, defines a Lévy process.

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Remark 3.13 We have shown, without studying this process, that the càdl'ag version of a Brwonain motion is a.s. continuous! In effect, $\psi(u) = -\frac{1}{2}u^2$ defines a Lévy process in in the sense of Definition 1.1 because for each t, $e^{-t\frac{u^2}{2}}$ is positive definite. We regularize, remove the jumps, and obtain a formula (B). Uniqueness show that $\nu(dx) \equiv 0$, a = 0, and thus there are no discontinuities!

Notes

- ¹I.e. for all u_1, \ldots, u_k the $k \times k$ -matrix $(\varphi_t(u_i u_j))_{i,j}$ is non-negative definite.
- ² Because then $X_s \stackrel{d}{\to} 0$, and sequence of random variables which converges in distribution to a constant also converges in probability to that constant.
 - 3 Every martingale has a càdlàg modification.
- $a \mapsto \frac{e^{i\langle u, X_s(\omega)\rangle}}{\varphi_s(u)}$ has unique left– and right limits as $s \uparrow t$ or $s \downarrow t$ along S. If, e.g. we two have sequences $s_n \uparrow t$ and $s'_n \uparrow t$ such that $X_{s_n}(\omega \to a \text{ and } X_{s'_n}(\omega) \to b$, then $\frac{e^{i\langle u, X_{s_n}(\omega)\rangle}}{\varphi_{s_n}(u)} \to \frac{e^{i\langle u, X_{s_n}(\omega)\rangle}}{\varphi_{t}(u)}$ and $\frac{e^{i\langle u, X_{s_n}(\omega)\rangle}}{\varphi_{s'_n}(u)} \to \frac{e^{i\langle u, X_{s_n}(\omega)\rangle}}{\varphi_{t}(u)}$ so that $e^{i\langle u,a\rangle} = e^{i\langle u,b\rangle}$.
 - ⁵ If $\mathbb{E}[e^{i\langle u,X\rangle}]=1$ for all u, then X has law δ_0 by Lévy inversion. So $\mathbb{P}(X=0)=1$.
- ⁶ We use here the Lévy-Doob Downward Theorem: If Z is an integrable R.V. and $\mathcal{G}_n \downarrow \mathcal{G}$, then $\mathbb{E}[Z|\mathcal{G}_n] \rightarrow \mathbb{E}[Z|\mathcal{G}_n]$
- We see that $\mathbb{E}[e^{i\sum_{j}\langle u_{j}X_{s_{j}}\rangle}|\mathcal{F}_{t}] = \mathbb{E}[e^{i\sum_{j}\langle u_{j}X_{s_{j}}\rangle}|\mathcal{F}_{t+}]$ a.s. for all u_{j}, s_{j} . Now the collection $\mathcal{M} := \{e^{i\sum_{j}\langle u_{j}X_{s_{j}}\rangle}: e^{i\sum_{j}\langle u_{j}X_{s_{j}}\rangle: e^{i\sum_{j}\langle u_{j}X_{s_{j}}\rangle: e$ u_i, s_i is closed under multiplication and conjugation. If $\mathcal{H} := \{Z : \mathbb{E}[Z|\mathcal{F}_{t+}] = \mathbb{E}[Z|\mathcal{F}_t]\}$, then \mathcal{H} is a vector space closed under bounded pointwise convergence. Hence \mathcal{H} contains every $\sigma(\mathcal{M})$ -measurable function — this is the monotone class theorem for complex-valued functions — cf. Bichteler Stochastic Integration with Jumps, Exercise A.3.5.

$$\mathbb{E}\left[I_{A}e^{i\sum_{j=1}^{m}\langle u_{j},X_{T+t_{j}}-X_{T+t_{j-1}}\rangle}\right] = \mathbb{E}\left[I_{A}\prod_{j=1}^{m}\frac{M_{T+t_{j}}^{u_{j}}}{M_{T+t_{j-1}}^{u_{j}}}\frac{\varphi_{T+t_{j}}}{\varphi_{T+t_{j-1}}}\right]$$

$$= \mathbb{E}\left[I_{A}\prod_{j=1}^{m-1}\frac{M_{T+t_{j}}^{u_{j}}}{M_{T+t_{j-1}}^{u_{j}}}\frac{\varphi_{T+t_{j}}}{\varphi_{T+t_{j-1}}}\mathbb{E}\left[\frac{M_{T+t_{m}}^{u_{m}}}{M_{T+t_{m-1}}^{u_{m}}}\frac{\varphi_{T+t_{m-1}}}{\varphi_{T+t_{m-1}}}|\mathcal{F}_{T+t_{m-1}}\right]\right]$$

Now
$$\frac{\varphi_{T+t_m}}{\varphi_{T+t_{m-1}}} = \varphi_{t_m-t_{m-1}}(u_m)$$
 and $\mathbb{E}\left[\frac{M_{T+t_m}^{u_m}}{M_{T+t_{m-1}}^{u_m}} | \mathcal{F}_{T+t_{m-1}}\right] = 1$. Apply m times.

⁹ With $A = \Omega$, the formula shows that the law of $(X_t)_t$ and $(X_{T+t} - X_T)_t$ are the same. ¹⁰Choose b s.t. $0 < b < -\frac{\ln a}{2M}$. Then $e^{2bM}a < 1$, so $\sum_n (e^{2bM}a)^n < \infty$. Now

$$\mathbb{E}[e^{b|X_t|}] = \sum_{n} \mathbb{E}[e^{b|X_t|} \Big| 2(n-1)M < |X_t| \le 2nM] \mathbb{P}(2(n-1)M < |X_t| \le 2nM)$$
$$\le \sum_{n} e^{2nMb} a^{n-1} = \frac{1}{a} \sum_{n} (e^{2bM} a)^n < \infty$$

Finally, $\mathbb{E}[e^{b|X_t|}] \geq \frac{b^n|X_t|^n}{n!}$, so $\mathbb{E}[|X_t|^n] < \infty$.

By induction, T_n has density function $t \mapsto \frac{a^n t^{n-1}}{(n-1)!} e^{-at}$, so

$$\mathbb{P}(T_{n+1} > t, T_n \le t) = \int_0^t \mathbb{P}(T_{n+1} - T_n | t - s | T_n = s) \frac{a^n s^{n-1}}{(n-1)!} e^{-as} ds = \int_0^t e^{-a(t-s)} \frac{a^n s^{n-1}}{(n-1)!} e^{-as} ds = \frac{a^n t^n}{n!} e^{-at} ds$$

$$^{12} \sum_{j,k} \lambda_j \bar{\lambda_k} e^{-at(1 - e^{i(u_j - u_k)})} = e^{-at} |\sum_j \lambda_j e^{iu_j}|^2.$$

- ¹³ As on p.1, Kolmogorov's theorem guarantees the existence of a stochastic process with the correct law, and Theorem 1.2 guarantees the existence of a càdlàg version thereof
- ¹⁴ Using the characteristic function $\mathbb{E}[e^{iuN_t}] = e^{-at(1-e^{iu})}$, and differentiating w.r.t. u, it is easy to see that $\mathbb{E}[N_t] = at = \operatorname{Var}(N_t).$
 - ¹⁵Doob's L^2 -inequality. Moreover the product of two L^2 -variables is in L^1 , by Hölder's inequality.
- ¹⁶ It follows from the theory of capacities and analytic sets that if σ is a stopping time, B a Borel set and Y a progressively measurable process, then $\tau := \inf\{t > \sigma : Y_t \in B\}$ is a stopping time, provided that the filtration satisfies the usual conditions. Now if X is càdlàg adapted, then $\Delta X := X - X_{-}$ is progressively measurable.
- ¹⁷ f has a discontinuity of the second kind at t if one of the limits $\lim_{s \uparrow t} f(s)$, $\lim_{s \downarrow t} f(s)$ does not exist. In this case, there would be infinitely many jumps of amplitude $> \varepsilon$ by time t, for some $\varepsilon > 0$.
 - ¹⁸ If N is a Poisson process with parameter a, then $\mathbb{E}[N_1] = a$; cf. footnote 14.

¹⁹Note that if $\sum_{j} I_{B_{j}} = I_{B}$, then $f(\Delta X_{S_{B}^{n}(\omega)})I_{\{S_{B}^{n}(\omega) \leq t\}} = f(\Delta X_{S_{B_{j}}^{m}(\omega)})I_{\{S_{B_{j}}^{m}(\omega) \leq t\}}$ for some j, m, and conversely. Hence $\sum_{n\geq 1} f(\Delta X_{S_B^n}) I_{\{S_B^n \leq t\}} = \sum_{n\geq 1} \sum_j f(\Delta X_{S_B^n}) I_{\{S_{B_j}^n \leq t\}}$. Now if $f = \sum_j a_j I_{B_j}$, then $\int_B f(x) N_t(dx) = \sum_j f(\Delta X_{S_B^n}) I_{\{S_B^n \leq t\}} = \sum_j f(\Delta X_{S_B^n}) I_{\{S_B^n \leq t$ $\sum_{j} a_{j} N_{t}(B_{j}) = \sum_{j} a_{j} \sum_{n \geq 1} I_{\{S_{B_{j}}^{n} \leq t\}} = \sum_{n \geq 1} \sum_{j} a_{j} I_{\{S_{B_{j}}^{n} \leq t\}} = \sum_{n \geq 1} \sum_{j} f(\Delta X_{S_{B_{j}}^{n}}) I_{\{S_{B_{j}}^{n} \leq t\}} = \sum_{n \geq 1} f(\Delta X_{S_{B}^{n}}) I_{\{S_{B_{j}}^{n} \leq t\}}.$

Since X is càdlàg, $X_u \stackrel{\text{a.s.}}{\to} 0$, as $u \downarrow 0$, and hence also $X_u \stackrel{\mathbb{P}}{\to} 0$.

The process \hat{Z} defined by $\hat{Z}_s := Z_{t+s} - Z_t$ has the same law as Z, by the Strong Markov property. E.g. $\hat{X}_s(B)$ is number of jumps X has in B between times t and s+t, and this has the same law as $X_s(B)$.

²³Because $\nu(B) := \mathbb{E}[N_1(B)]$, it follows that $\mathbb{E}[N_t(B)] = t\nu(B)$: By stationary independent increments it is clear that for natural numbers p,q we have $\mathbb{E}[N_p(B)] = p\nu(B)$, and that $\mathbb{E}[N_p(B)] = q\mathbb{E}[N_{\frac{p}{q}}(B)]$, so that $\mathbb{E}[N_{\frac{p}{q}}(B)] = \frac{p}{q}\nu(B)$ for any non-negative rational $\frac{p}{q}$. Now by the càdlàg property, $N_t(B) = \lim_{r \in \mathbb{Q}} N_r(B)$, so

dominated convergence yields $\mathbb{E}[N_t(B)] = t\nu(B)$.

24 Observe that $\mathbb{E}\left[\left(\sum_j a_j(N_t(B_j) - t\nu(B_j)\right)^2\right] = \mathbb{E}\left[\left(\sum_j \hat{N}_t(B_j)\right)^2\right] = \sum_j a_j^2 \mathbb{E}[\hat{N}_t(B_j)^2] = \sum_j a_j^2 t\nu(B_j)$, using footnote 14.

²⁵Choose a sequence f_n of step functions so that $|f_n - f| \leq 2^{-n}$. Put $Z := \int_B f(x) \ N_t(dx)$, $Z_n := \int_B f_n(x) \ N_t(dx)$. By dominated convergence, we have $|Z_n(\omega) - Z(\omega)| \leq \int_B |f_n(x) - f(x)| \ N_t(dx, \omega) \leq 2^{-n} N_t(B) \to 2^{-n} N_t(B)$ 0, so by dominated convergence $\mathbb{E}[Z_n] \to \mathbb{E}[Z]$. Similarly, $\int_B f_n(x) \nu(dx) \to \int f(x) \nu(dx)$. Since $\mathbb{E}[Z_n] = \int_{\mathbb{R}} f(x) \nu(dx) = \int_{\mathbb{R}} f(x) \nu(dx)$. $\begin{array}{l} \text{E}[Z_n] = t \int_B f_n(x) \ \nu(dx) \to t \int_B f(x) \ \nu(dx), \text{ we obtain } \mathbb{E}[Z] = t \int_B f(x) \ \nu(dx). \text{ Similarly, } let Y := \int_B f(x) \ N_t(dx) - t \int_B f(x) \ \nu(dx), Y_n := \int_B f_n(x) \ N_t(dx) - t \int_B f_n(x) \ \nu(dx). \text{ Then } |Y_n - Y| \le 2^{-n} (N_t(B) + t\nu(B)), \text{ so by dominated convergence } \mathbb{E}[|Y_n - Y|^2] \to 0, \text{ so } \mathbb{E}[Y_n^2] \to \mathbb{E}[Y^2]. \text{ Similarly, } \int_B f_n(x)^2 \ \nu(dx) \to \int_B f(x)^2 \ \nu(dx). \text{ Now } \mathbb{E}[Y^2] = t \int_B f_n(x)^2 \ \nu(dx) \to t \int_B f(x)^2 \ \nu(dx). \end{array}$

 $^{27}\mathrm{A}$ direct consequence of Lemma 3.5.

²⁸Suppose that $M_n := M_{f_n I_B}$ for some sequence $f_n \in L^2(d\nu)$, and that $(M_n)_n$ is a Cauchy sequence in \mathcal{H}_B . Then by (\star) , $(f_n I_B)_n$ is Cauchy in $L^2(d\nu)$ and thus converges to some $f = f I_B$. By (\star) again, $M_n \to M_{f I_B}$ in

 \mathcal{H}_B .

29 The compensated Poisson processes $(\hat{N}_t(A))_t$, where A is a Borel subset of B, generate \mathcal{H}_B : Each $fI_B \in \mathcal{H}_B$. Then $f \in \mathcal{H}_B$ then f(x) = f(x) = f(x) and f(x) = f(x) = f(x). $L^2(d\nu)$ is a limit of step functions $\sum_i a_i I_{A_i}$, where $A_i \subseteq B$. Then $\int_B f N_t(dx) - t \int_B f(x) \nu(dx)$ is a limit of $\sum_{i} a_{j} N_{t}(A_{j}).$

 30 Recall a result of Kač, which states that two random variables Y, Z are independent iff $\mathbb{E}[e^{i\langle u, Y \rangle + i\langle v, Z \rangle}] =$ $\mathbb{E}[e^{i\langle u,Y\rangle}]\mathbb{E}[e^{i\langle v,Z\rangle}]$ for all u,v.

Using independence, $q_t(\sum_{m < k \le n} Y_t(B_k)) = \sum_{m < k \le n} \mathbb{E}[Y_t(B_k)^2] = t \int_{\bigcup_{m < k \le n} B_k} |x|^2 \nu(dx) \le \frac{t}{m^2} \nu(\bigcup_{m < k \le n} B_k) \to 0$ as $m \to \infty$. Hence the (sequence $(Y(A_n))_n$ corresponding to the) series $\sum_{k=1}^{\infty} Y(B_k)$ is Cauchy in \mathcal{M} , and thus converges to some $X^d \in \mathcal{M}$. Moreover, using dominated convergence and the fact that the $Y(A_n)$ are Lévy processes, we obtain (i): $\mathbb{E}[e^{i\langle u, X_{t+s}^d - X_t^d \rangle} | \mathcal{F}_t] = \lim_n \mathbb{E}[e^{i\langle u, Y_{t+s}(A_n) - Y_t(A_n) \rangle} | \mathcal{F}_t] = \lim_n \mathbb{E}[e^{i\langle u, Y_{t+s}(A_n) - Y_t(A_n) \rangle}] = \lim_n \mathbb{E}[e^{i\langle u, Y_{t+s}(A_n) - Y_t(A_n) - Y_t(A_n) \rangle}] = \lim_n \mathbb{E}[e^{i\langle u, Y_{t+s}(A_n) - Y_t(A_n) - Y_t(A_n) - Y_t(A_n) - Y_t(A_n)]}] = \lim_n \mathbb{E}[e^{i\langle u, Y_{t+s}(A_n) - Y_t(A_n) - Y_t(A_n) - Y_t(A_n)]}] = \lim_n \mathbb{E}[e^{i\langle u, Y_{t+s}(A_n) - Y_t(A_n) - Y_t(A_n) - Y_t(A_n)]}] = \lim_n \mathbb{E}[e^{i\langle u, Y_{t+s}(A_n) - Y_t(A_n) - Y_t(A_n) - Y_t(A_n)]}] = \lim_n \mathbb{E}[e^{i\langle u, Y_{t+s}(A_n) - Y_t(A_n) - Y_t(A_n) - Y_t(A_n)]}]$ $\mathbb{E}[e^{i\langle u, X_{t+s}^d - X_t^d \rangle}], \text{ which shows that } X^d \text{ has independent increments, and (ii): } \mathbb{E}[e^{i\langle u, X_{t+s}^d - X_t^d \rangle}] = \lim_n \mathbb{E}[e^{i\langle u, Y_{t+s}(A_n) - Y_t(A_n) \rangle}] = \mathbb{E}[e^{i\langle u, X_{t+s}^d - X_t^d \rangle}]$ $\lim_n \mathbb{E}[e^{i\langle u, Y_s(A_n)\rangle}] = \mathbb{E}[e^{i\langle u, X_s^d\rangle}], \text{ which shows that } X^d \text{ has independent increments.}$ $^{32}\text{Note that } Y(A_n) = \int_{\{\frac{1}{n+1} < |x| \le 1\}} |x| \left(N_t(dx) - t\nu(dx)\right), \text{ and that } X^d = \lim_n Y(A_n)$

³³We start with three observations. (i): First observe that if $(X_t)_t$ is an *n*-dimensional Lévy process and A is a $m \times n$ -matrix, then AX is an m-dimensional Lévy process: For $\mathbb{E}[e^{i\langle u, AX_{t+s} - AX_t \rangle} | \mathcal{F}_t] = \mathbb{E}[e^{i\langle A^\top u, X_{t+s} - X_t \rangle} | \mathcal{F}_t] = \mathbb{E}[e^{i\langle A^\top u, X_{t+s} - X_t \rangle} | \mathcal{F}_t] = \mathbb{E}[e^{i\langle A^\top u, X_{t+s} - X_t \rangle} | \mathcal{F}_t]$ $\mathbb{E}[e^{i\langle A^{\top}u,X_s\rangle}] = \mathbb{E}[e^{i\langle u,AX_s\rangle}]$ shows that AX has independent stationary increments. (ii): Next recall that an n-dimensional random vector Y is multivariate Gaussian if and only if each linear combination $\lambda^{\top}Y$ is univariate Gaussian, for any $\lambda \in \mathbb{R}^n$. (iii): Finally, if $(Z_t)_t$ is a n-dimensional Lévy process such that each random vector Z is multivariate Gaussian, then $(Z_t)_t$ is a Gaussian process: For each linear combination $\sum_{j=1}^m \lambda_k Z_{t_k}$ can be written as a linear combination of independent normally distributed random vectors $\sum_{j=1}^{m} \lambda_k Z_{t_k} = \sum_{k=1}^{m} \gamma_k (Z_{t_k} - Z_{t_{k-1}})$ (where $\gamma_k := \sum_{j=k}^{m} \lambda_j$ and $t_0 := 0$), so that each linear combination $\sum_{j=1}^{m} \lambda_k Z_{t_k}$ is Gaussian, i.e. $(Z_t)_t$ is a Gaussian process. Having made these three observations, we now proceed: Suppose that $X = (X_t)_t$ is an ndimensional Lévy process with continuous sample paths. Then each linear combination $\lambda^{\top}X$ is a one–dimensional Lévy process (by (i)) with continuous sample paths. If we can show that, for each t and λ , the random variable

 $\lambda^{\top} X_t$ is Gaussian, then by (ii) each random vector X_t is multivariate Gaussian, so by (iii) $(X_t)_t$ is a Gaussian process. Thus it suffices to show that whenever B is a one-dimensional Lévy process with continuous sample paths, then each random variable B_t is Gaussian — just take $B_t = \lambda^{\top} X_t$.

 34 Here is another proof of Lemma 3.11, which uses the Lévy characterization of Brownian motion. Suppose that X_t is a d-dimensional Lévy process with continuous sample paths. Then it has moments of all orders, by Corollary 1.5. In particular, the process $X_t - \mathbb{E}[X_t]$ is a continuous martingale centered at 0. We now show that any centered continuous Lévy process is a Brownian motion in the loose sense: Components need not be independent, but are multi-variate Gaussian.

So let $X_t = (X_t^{(1)}, \dots, X_t^{(d)})$ be a centered Lévy process with continuous sample paths. Let A be the non–negative definite symmetric $d \times d$ –matrix defined by

$$A_{ij} = \mathbb{E}[X_1^{(i)} X_1^{(j)}]$$

where $X_t = (X_t^{(1)}, \dots, X_t^{(d)})$. We claim that the quadratic covariation process of $X_t^{(i)}$ and $X_t^{(j)}$ is given by

$$[X^{(i)}, X^{(j)}]_t = A_{ij}t$$

Recall that $\mathbb{E}[e^{i\langle u, X_t \rangle}] = e^{-t\psi(u)}$ for some $\psi : \mathbb{R}^d \to \mathbb{C}$ with $\psi(0) = 0$. Since X_t has moments of all orders, the function ψ is \mathcal{C}^{∞} . Moreover, since $\mathbb{E}[X_t^{(i)}] = 0$, we have $\frac{\partial}{\partial u^{(i)}}\Big|_{u=0} e^{-t\psi(u)} = 0$, and thus that $\frac{\partial}{\partial u^{(i)}}\Big|_{u=0} \psi(u) = 0$ also. It now follows easily that

$$\mathbb{E}[X_t^{(i)}X_t^{(j)}] = -\frac{\partial^2}{\partial u^{(i)}\partial u^{(j)}}\bigg|_{u=0}e^{-t\psi(u)} = -t\left.\frac{\partial^2}{\partial u^{(i)}\partial u^{(j)}}\right|_{u=0}\psi(u) = t\,\mathbb{E}[X_1^{(i)}X_1^{(j)}]$$

i.e. that

$$\mathbb{E}[X_t^{(i)}X_t^{(j)}] = A_{ij}t$$

To show that $[X^{(i)}, X^{(j)}]_t = A_{ij}t$, it suffices to show that $X_t^{(i)}X_t^{(j)} - A_{ij}t$ is a martingale. By the fact that increments are independent with mean zero, we have

$$\begin{split} & \mathbb{E}[X_t^{(i)}X_t^{(j)} - X_s^{(i)}X_s^{(j)}|\mathcal{F}_s] \\ &= \mathbb{E}[(X_t^{(i)} - X_s^{(i)})(X_t^{(j)} - X_s^{(j)}) + X_s^{(i)}(X_t^{(j)} - X_s^{(j)}) + X_s^{(j)}(X_t^{(i)} - X_s^{(i)})|\mathcal{F}_s] \\ &= \mathbb{E}[(X_t^{(i)} - X_s^{(i)})(X_t^{(j)} - X_s^{(j)})] \end{split}$$

Taking expectations on both sides shows that

$$\mathbb{E}[X_t^{(i)}X_t^{(j)} - X_s^{(i)}X_s^{(j)}] = \mathbb{E}[(X_t^{(i)} - X_s^{(i)})(X_t^{(j)} - X_s^{(j)})]$$

It follows that

$$\mathbb{E}[X_t^{(i)}X_t^{(j)} - X_s^{(i)}X_s^{(j)}|\mathcal{F}_s] = \mathbb{E}[X_t^{(i)}X_t^{(j)} - X_s^{(i)}X_s^{(j)}] = A_{ij}(t-s)$$

and thus that $X_t^{(i)}X_t^{(j)} - A_{ij}t$ is a martingale.

Now, for $\lambda \in \mathbb{R}^d$, let $Z_t^{\lambda} = \langle \lambda, X_t \rangle$. Then Z_t^{λ} is a centered continuous one–dimensional martingale. Using the fact that the covariance process bracket operation is bilinear, we see that the quadratic variation of Z_t^{λ} is given by

$$[Z^{\lambda}]_t = \langle \lambda, A\lambda \rangle \ t$$

Hence, by Lévy's characterization, Z_t^{λ} is a Brownian motion with variance parameter $\langle \lambda, A\lambda \rangle$ (i.e. $Z_t^{\lambda} \sim N(0, \langle \lambda, A\lambda \rangle t)$). It now follows that

$$\mathbb{E}[e^{i\langle u, X_t \rangle}] = \varphi_{Z_t^u}(1) = e^{-\frac{1}{2}\langle u, Au \rangle t}$$

which proves that X_t is a d-dimensional Brownian motion with covariance matrix A.

Now if X is a Lévy process with continuous sample paths, then $X_t - \mathbb{E}[X_t]$ is centered, and thus a Brownian motion. If we define $\gamma = \mathbb{E}[X_1]$, then $\mathbb{E}[X_t] = \gamma t$, and so $\mathbb{E}[e^{i\langle u, X_t - \gamma t \rangle}] = e^{-\frac{1}{2}\langle u, Au \rangle t}$ for some symmetric non–negative definite matrix A. We have proved:

Theorem 3.14 Suppose that X_t is a Lévy process with continuous sample paths. Then there exists $\gamma \in \mathbb{R}^d$ and a symmetric non-negative definite $d \times d$ -matrix A such that

$$\mathbb{E}[e^{i\langle u, X_t \rangle}] = e^{i\langle u, \gamma \rangle \ t - \frac{1}{2}\langle u, Au \rangle \ t}$$

Hence X_t is a d-dimensional arithmetic Brownian motion with drift γ and covariance matrix A.

³⁵For $n \ge 1$, we have $\mathbb{E}[B_t^n] = i^{-n} \frac{d^n}{du^n}|_{u=0} e^{-t\psi(u)} = \sum_{k=1}^n a_k t^k$. Here the t^n -term will have coefficient $a_n = (-\psi'(0))^n = 0$.

³⁶Observe that

$$\left| \frac{1}{2} u^2 \sum_{j} \mathbb{E} \left[(\Delta B_j)^2 \cdot (e^{iu(B_{t_j} + \theta_j \Delta B_j)} - e^{iuB_{t_j}}) \right] \right| \leq \frac{1}{2} |u|^2 \int \sum_{j} \Delta B_j^2 |(e^{iu(B_{t_j} + \theta_j \Delta B_j)} - e^{iuB_{t_j}}) d\mathbb{P}$$

$$\leq \frac{1}{2} |u|^2 \int_{A_{\alpha}} \sum_{j} \Delta B_j^2 |u\theta_j \Delta B_j| d\mathbb{P} + \frac{1}{2} |u|^2 \int_{A_{\alpha}^c} \sum_{j} \Delta B_j^2 \cdot 2 d\mathbb{P}$$

using the facts that $|e^{ih}| \leq |h|$ and that $|e^{ih} - 1| \leq 2$. Now $|u\theta_j \Delta B_j| \leq \alpha |u|$ on A_α $^{37} \int_{A_\alpha^c} \sum_j \Delta B_j^2 \ d\mathbb{P} = \mathbb{E}[I_{A_\alpha^c} \sum_j \Delta B_j^2] \leq \sqrt{\mathbb{E}[I_{A_\alpha^c}^2]} \sqrt{\mathbb{E}[(\sum_j \Delta B_j^2)^2]}$, by the Cauchy-Schwarz or Hölder inequality. $^{38} - \frac{1}{2}u^2 \int_0^t e^{-s\psi(u)} ds = \frac{u^2}{2\psi(u)} (e^{-t\psi(u)} - 1).$

 39 This is the Lévy–Khintchine formula for the characteristic function of an infinitely divisible distribution.

⁴⁰i.e. $Q(u) = u^{\top} \Sigma u$ for some symmetric positive definite matrix Σ . Here, Σ is the covariance matrix for the continuous Gaussian part.

⁴¹It is useful to observe that for any $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$, we have $|e^{i\theta} - \sum_{k=1}^{n-1} \frac{(i\theta)^k}{k!}| \leq \frac{|\theta|^n}{n!}$. This follows immediately from the identity $e^{i\theta} = \sum_{k=1}^{n-1} \frac{(i\theta)^k}{k!} + \frac{i^n}{(n-1)!} \int_0^{\theta} (\theta - x)^{n-1} e^{ix} dx$, which may be proved using induction and integration by parts.

Recall that $|x|^2 \wedge 1$ in ν -integrable. On $\{|x| \geq 1\}$ we have $\frac{1-e^{i\langle sy,x\rangle}}{s^2} \leq \frac{2}{s^2} \leq 2$ for s sufficiently large, and $2I_{\{|x|\geq 1\}}$ is ν -integrable. On $\{|x|<1\}$ we have $\frac{1-e^{i\langle sy,x\rangle}-i\langle sy,x\rangle}{s^2} \leq \frac{|\langle sy,x\rangle|^2}{2s^2} \leq |x|^2$, and $|x|^2I_{\{|x|<1\}}$ is ν -integrable. In both cases, we have domination by a ν -integrable function.

⁴³ For the one–dimensional case, we have

$$\begin{split} &\psi(u) - \frac{1}{2} \int_{u-1}^{u+1} \psi(v) - \frac{1}{2} Q(v) \ dv \\ &= -iau + \int_{\mathbb{R}} 1 - e^{iux} - iux I_{\{|x| < 1\}} \nu(dx) - \frac{1}{2} \int_{u-1}^{u+1} -iav + \int_{\mathbb{R}} 1 - e^{ivx} - ivx I_{\{|x| < 1\}} \nu(dx) \ dv \\ &= \frac{1}{2} \left[\int_{-1}^{1} -iau + \int_{\mathbb{R}} 1 - e^{iux} - iux I_{\{|x| < 1\}} \nu(dx) \ dv - \int_{-1}^{1} -ia(u+v) + \int_{\mathbb{R}} 1 - e^{i(u+v)x} - i(u+v)x I_{\{|x| < 1\}} \nu(dx) \ dv \right] \\ &= \frac{1}{2} \left[\int_{-1}^{1} aiv + \int_{\mathbb{R}} -(e^{iux} - e^{i(u+v)x}) + (iux - i(u+v)x) I_{\{|x| < 1\}} \nu(dx) \ dv \right] \\ &= -\frac{1}{2} \left[\int_{-1}^{1} \int_{\mathbb{R}} e^{iux} (1 - e^{ivx}) + ivx I_{\{|x| < 1\}} \nu(dx) \ dv \right] \end{split}$$

But, using the inequality derived in footnote 41, $|e^{iux}(1-e^{ivx})+ivxI_{\{|x|<1\}}|$ is ≤ 2 on $\{|x|\geq 1\}$, and is $\leq |e^{iux}|\cdot|1-e^{ivx}+ivx|+|ivx|\cdot|1-e^{iux}|\leq \frac{1}{2}|v|^2|x|^2+|v||x||ux|=K|x|^2$ on $\{|x|<1\}$. Hence $e^{iux}(1-e^{ivx})+ivxI_{\{|x|<1\}}$ is $\nu(dx) \otimes dv$ -integrable, so Fubini's Theorem applies:

$$-\frac{1}{2} \int_{-1}^{1} \int_{\mathbb{R}} e^{iux} (1 - e^{ivx}) + ivx I_{\{|x| < 1\}} \nu(dx) dv$$

$$= -\frac{1}{2} \int_{\mathbb{R}} e^{iux} \int_{-1}^{1} 1 - e^{ivx} + ivx I_{\{|x| < 1\}} dv \nu(dx)$$

$$= -\int_{\mathbb{R}} e^{iux} \left(1 - \frac{\sin x}{x}\right) \nu(dx)$$

Hence $-\psi(u) + \frac{1}{2}Q(u)$ is the Fourier transform of the positive measure $\rho(dx) := (1 - \frac{\sin x}{x}) \nu(dx)$. By Fourier inversion, ρ is determined by ψ and Q. Since ψ determines Q, we see that ρ , hence ν , is determined by ψ .

⁴⁵Again, using the inequality of footnote 41, we have $|1-e^{i\langle u,x\rangle}| \le 2$ on $\{|x| \ge 1\}$ and $\le \frac{1}{2}|u|^2|x|^2$ on $\{|x| < 1\}$, so if $\int |x|^2 \wedge 1 \nu(dx)$ is finite, both integrals in (B) are defined.